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Statistical Properties of Orbit Perturbations Induced by the Earth's Anomalous Gravity

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Prepared for COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-1001.

This report, which documents research carried out from 15 October 1965 to 1 December 1966, was submitted on 10 April 1967 to Col William H. Weaver, Jr., SSD (SSON), for review and approval.

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

William H. Weaver, Jr.
Colonel, USAF
Director for Development
AF Satellite Control Facility

ABSTRACT

Lower bounds are obtained for the covariances of the orbital elements of an initially circular orbit due to omissions of harmonics of a given degree in the expansion of the earth's gravity potential. This is accomplished by applying the methods of linear estimation theory to the surface gravimetry results of Kaula, extended to satellite altitudes. Numerical results are presented for the case of a 100-n mi geocentric orbit.

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SECTION I

INTRODUCTION

With the increased accuracy requirements placed upon the investigator in the field of astrodynamics, it has become necessary to retain more and more coefficients in the expansion of the earth's gravity potential in terms of spherical harmonics. However, the inaccuracies inherent in omitting harmonics of a given degree have yet to be estimated. This analysis provides lower bounds for these effects for the class of initially circular geocentric orbits.

SECTION II

THE COVARIANCE EQUATIONS

It is assumed that, for small deviations from circularity, the elements of an initially circular geocentric orbit are governed by a linear system of differential equations of the form

$$\frac{d\,\Delta X(t)}{dt} = F(t)\,\Delta X(t) + G(t)\,u(t) \tag{1-a}$$

where

X = a 6×1 column vector representing the two body orbital elements

$u(t)$ = a 3×1 column vector of zero mean representing the forcing function

$F(t)$ and $G(t)$ = 6×6 and 6×3 matrices, respectively

Δ = a deviation from circular values

t = the time

The solution to this system of equations is

$$\Delta X(t) = \phi(t, t_o)\,\Delta X(t_o) + \int_{t_o}^{t} \phi(t, \tau)\, G(\tau)\, u(\tau)\, d\tau \tag{1-b}$$

where

o = a subscript notation indicating an epoch value

$\phi(t, \tau)$ = the 6×6 state transition matrix, which transforms perturbations in the orbital elements at time τ into perturbations at time t.

Now, since this analysis is limited to initially circular orbits, we have from the definition of Δ

$$\Delta X(t_o) = 0$$

Thus, Eq. (1-b) can be rewritten

$$\Delta X(t) = \int_{t_o}^{t} \phi(t, \tau)\, G(\tau)\, u(\tau)\, d\tau \qquad (1\text{-c})$$

Further, noting that u(t) is of zero mean, from Eq. (1-c)

$$\mathrm{COV}\left[\Delta X(t)\right] = \int_{t_o}^{t} \int_{t_o}^{t} \phi(t, \tau)\, G(\tau)\, E\left[u(\tau)\, u^T(\eta)\right] G^T(\eta)\, \phi^T(t, \eta)\, d\tau\, d\eta \qquad (2)$$

where

E = an expected value
COV = the covariance

The quantity X is now equated to the following set of orbital elements (see Fig. 1):

ρ = radial separation between the satellite and geocenter

v = speed of the satellite

β = angle between the radius vector to the satellite and the velocity vector

i = the inclination

Ω = longitude of the ascending node

ζ = angle in the orbit plane between the point of maximum declination and the instantaneous satellite location (positive in the direction of satellite motion) i. e., the argument of latitude minus $\pi/2$ radians.

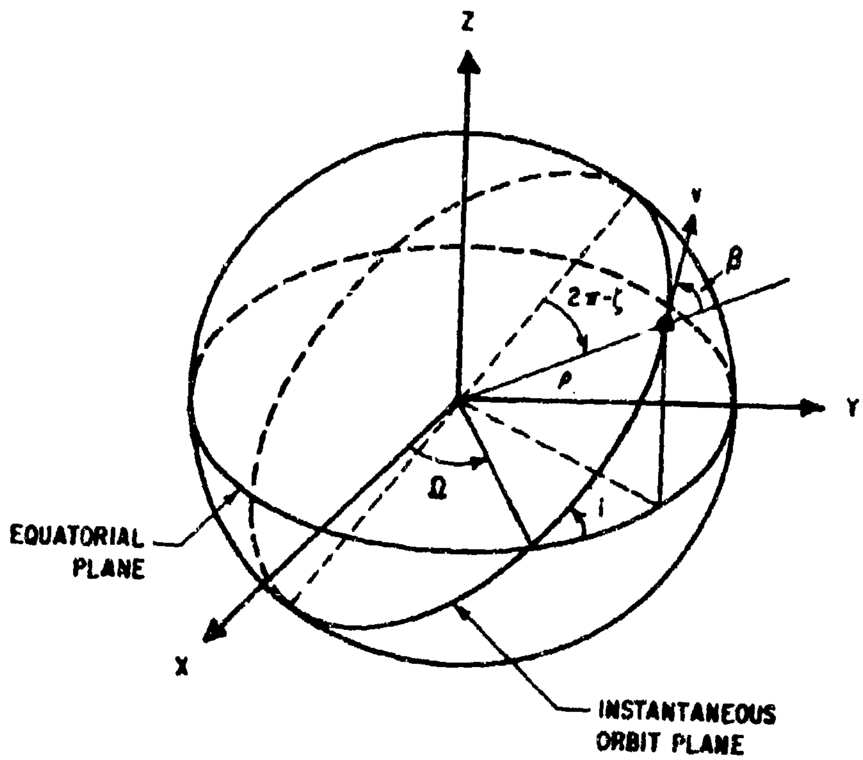

Figure 1. Orbital Elements

It is convenient to replace time by ζ as the independent variable (and conversely to replace ζ by t as an orbital element) thus modifying Eq. (2) to

$$\mathrm{COV}\left[\Delta X(\zeta)\right] = \int_{\zeta_o}^{\zeta} \int_{\zeta_o}^{\zeta} \phi(\zeta, \tau)\, G(\tau)\, E\left[u(\tau)\, u^T(\eta)\right] G^T(\eta)\, \phi^T(\zeta, \eta)\, d\tau\, d\eta \tag{3}$$

Prior to direct integration, expressions for the terms appearing in Eq. (3) as functions of the orbital elements and the components of the forcing function must be found. The analysis to accomplish this purpose follows.

SECTION III

EVALUATION OF TERMS APPEARING IN THE COVARIANCE INTEGRALS

Neglecting second and higher order terms in ΔX yields in the absence of the forcing function (Ref. 1)

$$\left(\frac{d\Delta\rho}{d\zeta}\right)_1 = \rho_o \Delta\beta$$

$$\left(\frac{d\Delta v}{d\zeta}\right)_1 = v_o \Delta\beta$$

$$\left(\frac{d\Delta\beta}{d\zeta}\right)_1 = -\frac{1}{\rho_o}\Delta\rho - \frac{2}{v_o}\Delta v \tag{4}$$

$$\left(\frac{d\Delta i}{d\zeta}\right)_1 = 0$$

$$\left(\frac{d\Delta\Omega}{d\zeta}\right)_1 = 0$$

$$\left(\frac{d\Delta t}{d\zeta}\right)_1 = \frac{1}{v_o}\Delta\rho - \frac{\rho_o}{v_o^2}\Delta v$$

Extending Eq. (4) to include the effects of the perturbing accelerations as given by the Gaussian equations of motion with ζ replacing time as independent variable (see e.g., Ref. 2) yields

$$
\begin{aligned}
\frac{d\Delta\rho}{d\zeta} &= -\rho_o \Delta\beta \\
\frac{d\Delta v}{d\zeta} &= v_o \Delta\beta + \frac{\rho_o}{v_o} S \\
\frac{d\Delta\beta}{d\zeta} &= \frac{1}{\rho_o}\Delta\rho - \frac{2}{v_o}\Delta v - \frac{\rho_o}{v_o^2} R \\
\frac{d\Delta i}{d\zeta} &= \frac{\rho_o \sin\zeta}{v_o^2} W \\
\frac{d\Delta\Omega}{d\zeta} &= \frac{\rho_o \cos\zeta}{v_o^2 \sin i} W \\
\frac{d\Delta t}{d\zeta} &= \frac{1}{v_o}\Delta\rho - \frac{\rho_o}{v_o^2}\Delta v + \frac{\rho_o^2 \cos\zeta \cos i}{v_o^3 \sin i} W
\end{aligned}
\qquad (5)
$$

where R, S, and W are the components of the perturbative acceleration; R lying in the direction of the increasing radius vector, W in the direction of the moment of momentum vector, and S in the instantaneous orbit plane with sense such that R, S, and W form an orthogonal right-handed set. It should be noted that Eq. (5) is equivalent to the matrix Eq. (1-a) with ζ replacing t as the independent variable and

$$
\Delta X = \begin{bmatrix} \Delta\rho \\ \Delta v \\ \Delta\beta \\ \Delta i \\ \Delta\Omega \\ \Delta t \end{bmatrix} \qquad (6\text{-a})
$$

$$
F = \begin{bmatrix} 0 & 0 & -\rho_o & 0 & 0 & 0 \\ 0 & 0 & v_o & 0 & 0 & 0 \\ -\frac{1}{\rho_o} & -\frac{2}{v_o} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{v_o} & -\frac{\rho_o}{v_o^2} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6-b}
$$

$$
u = \begin{bmatrix} R \\ S \\ W \end{bmatrix} \tag{6-c}
$$

$$
G(\tau) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\rho_o}{v_o} & 0 \\ -\frac{\rho_o}{v_o^2} & 0 & 0 \\ 0 & 0 & -\frac{\rho_o \sin\tau}{v_o^2} \\ 0 & 0 & \frac{\rho_o \cos\tau}{v_o^2 \sin i} \\ 0 & 0 & \frac{\rho_o^2 \cos\tau \cos i}{v_o^3 \sin i} \end{bmatrix} \tag{6-d}
$$

Since all of the coefficients appearing in Eq. (4) [or Eq. (6-b)] are constants, it is a straightforward matter to solve this system of equations for the transition matrix yielding

$$\phi(\zeta, \tau) = \begin{Bmatrix} 2 - \cos(\zeta - \tau) & \frac{2\rho_o}{v_o}[1 - \cos(\zeta - \tau)] & -\rho_o \sin(\zeta - \tau) & 0 & 0 & 0 \\ \frac{v_o}{\rho_o}[\cos(\zeta - \tau) - 1] & 2\cos(\zeta - \tau) - 1 & v_o \sin(\zeta - \tau) & 0 & 0 & 0 \\ -\frac{1}{\rho_o}\sin(\zeta - \tau) & -\frac{2}{v_o}\sin(\zeta - \tau) & \cos(\zeta - \tau) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{v_o}[3(\zeta - \tau) - 2\sin(\zeta - \tau)] & \frac{\rho_o}{v_o^2}[3(\zeta - \tau) - 4\sin(\zeta - \tau)] & -\frac{2\rho_o}{v_o}[1 - \cos(\zeta - \tau)] & 0 & 0 & 1 \end{Bmatrix} \quad (6\text{-e})$$

The R, S, and W components of the correlation matrix of the forcing function, due to effects of harmonics of given degree, are derived in Appendix A. This is accomplished by extending the surface gravimetry results of Kaula (Ref. 3) to satellite altitudes. Examination of these results shows all three components to be linear combinations of cosines of integer multiples of the satellite separation angle. Further, assuming these components are not cross-correlated yields a correlation matrix with terms of the form

$$E\left[u(\tau)\, u^T(\eta)\right] = \begin{bmatrix} \alpha \cos \lambda_1(\tau - \eta) & 0 & 0 \\ 0 & \beta \cos \lambda_2(\tau - \eta) & 0 \\ 0 & 0 & \gamma \cos \lambda_3(\tau - \eta) \end{bmatrix} \tag{6-f}$$

where α, β, γ and the integers λ_1, λ_2, λ_3 are defined in Appendix A.

SECTION IV

EVALUATION OF THE COVARIANCE INTEGRALS

It should be observed that a difficulty associated with the limits placed upon Eq. (3) is the implicit assumption that the covariance of the orbital elements is correlated with a monotonically increasing separation angle ζ. Of course, separation central angles are never greater than π radians. For purposes of the present analysis, this difficulty is ameliorated by neglecting correlations beyond a preselected separation angle denoted by A (where $A \leq \pi$). This necessitates reformulating the right hand side of Eq. (3) as the three double integrals

$$\mathrm{COV}\left[\Delta X(\zeta)\right] = \int_{\zeta_o}^{\zeta_o + A} \int_{\zeta_o}^{\tau + A} I(\tau, \eta)\, d\tau\, d\eta$$

$$+ \int_{\zeta_o + A}^{\zeta - A} \int_{\tau - A}^{\tau + A} I(\tau, \eta)\, d\tau\, d\eta \qquad (7)$$

$$+ \int_{\zeta - A}^{\zeta} \int_{\tau - A}^{\zeta} I(\tau, \eta)\, d\tau\, d\eta$$

where

$I(\tau, \eta)$ = The integrand of Eq. (3) = $\phi(t, \tau)\, G(\tau)\, E\left[u(\tau)\, u^T(\eta)\right] G^T(\eta)\, \phi^T(t, \eta)$

The (shaded) area in the τ, η plane over which Eq. (7) is integrated is shown in Fig. 2. Of course, Eq. (7) holds only when $\zeta - \zeta_o \geq 2A$. It should also be

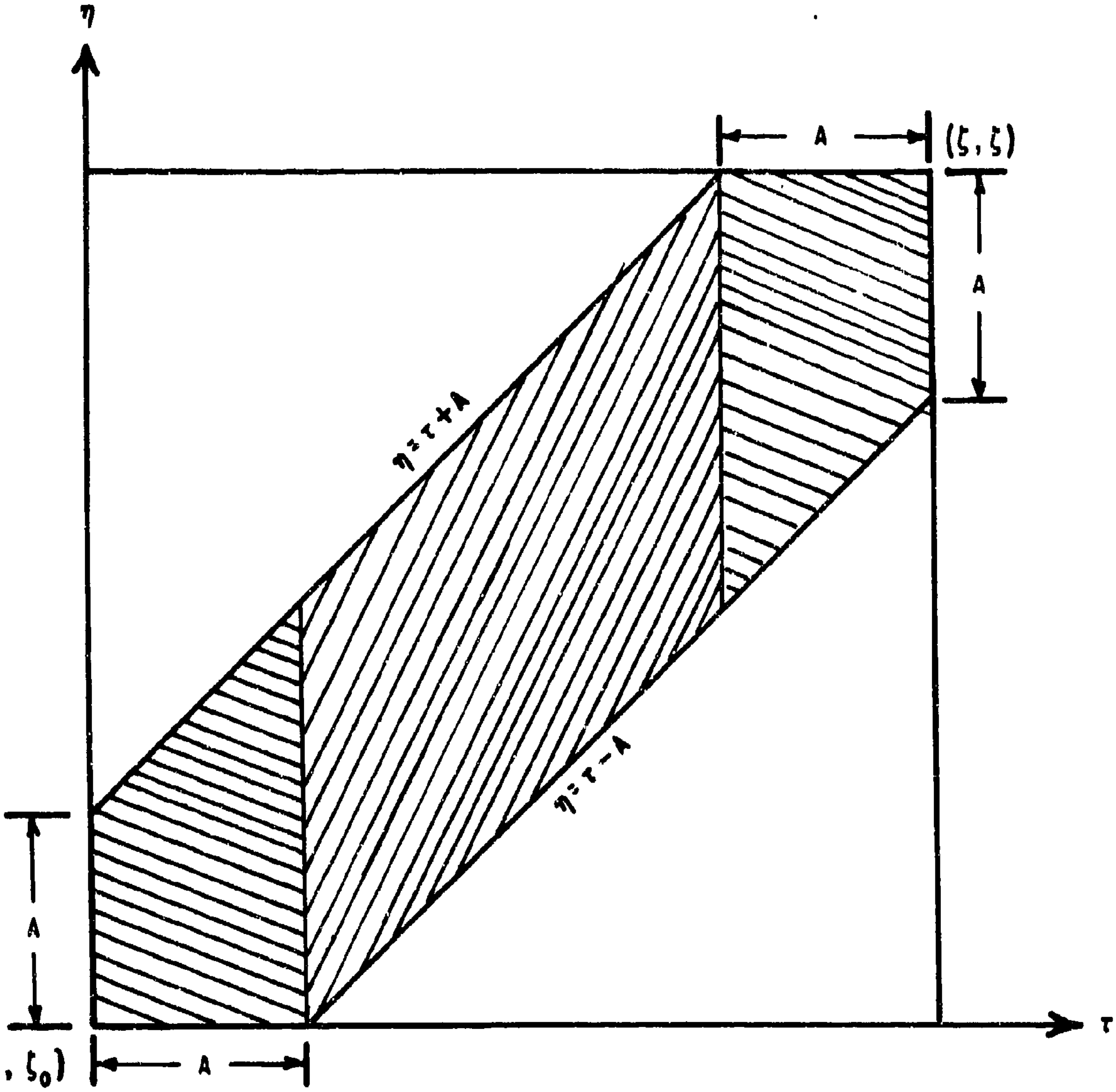

Figure 2. Regions of Integration for the Covariance Equation

noted that as formulated Eq. (7) yields only a lower bound for the covariance of the orbital elements since it completely neglects the covariance arising from re-crossings of previous ground traces, (i. e., it is equivalent to the assumption that the satellite passes over new topography with each revolution). Substituting Eqs. (6-d), (6-e), and (6-f) into Eq. (7) yields, after direct integration, the symmetrical matrix

$$\mathrm{COV}\begin{bmatrix}\Delta\rho\\ \Delta\nu\\ \Delta\beta\\ \Delta i\\ \Delta\Omega\\ \Delta t\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16}\\ a_{21} & a_{22} & a_{23} & 0 & 0 & a_{26}\\ a_{31} & a_{32} & a_{33} & 0 & 0 & a_{36}\\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46}\\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56}\\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}\end{bmatrix} \tag{8}$$

where the matrix elements are given in Appendix B.

SECTION V

NUMERICAL EXAMPLE

In order to perform the summation in Eqs. (A-14), (A-17), and (A-18), numerical values are needed for the degree variances σ_n^2 . For this purpose, the results obtained by Kaula in Refs. (3) and (4), for the degree variances of the terrestrial free-air anomalous gravity field degree variance, are adopted. These values are illustrated as a function of degree n in Fig. 3. Unfortunately, Kaula has only determined degree variances through n = 32. At n = 30 he obtained a negative degree variance, a result which is theoretically impossible as indicated by Eq. (A-13). Hence, for this analysis, degree variances beyond n = 30 were represented by $\sigma_n^2 = 35.09\,(0.949)^n$ (a relationship chosen so as to approximate degree variances below, and be well behaved above, n = 30). The solid line in Fig. 3 illustrates degree variances obtained from this relationship. Below n = 30, Kaula's degree variances were utilized. Further, correlations beyond separation angles of π radians were neglected, i.e., A was set equal to π.

The degree variances described above were then utilized to obtain numerical results for the diagonal terms of the matrix comprising the right hand side of Eq. (8). In Figs. 4 through 9, the graphs of the square root of these diagonal elements (representing the standard deviations of the orbital elements) are presented as a function of revolution for the case of a satellite in a circular, 100-n mi polar orbit. A small oscillatory component superimposed on the secular trend has been smoothed out of these figures. Each of these figures consists of three curves corresponding to three assumptions regarding the gravity model. The curve with the largest ordinates was obtained by summing from n = 2 to n = 50; that with the second largest ordinates by summing from n = 7 to n = 50; and that with the smallest ordinates by summing from n = 13 to n = 50. The exclusion of the contribution of the lower degrees terms in a spherical harmonic representation of the geopotential is equivalent to the assumption of a perfect knowledge of the

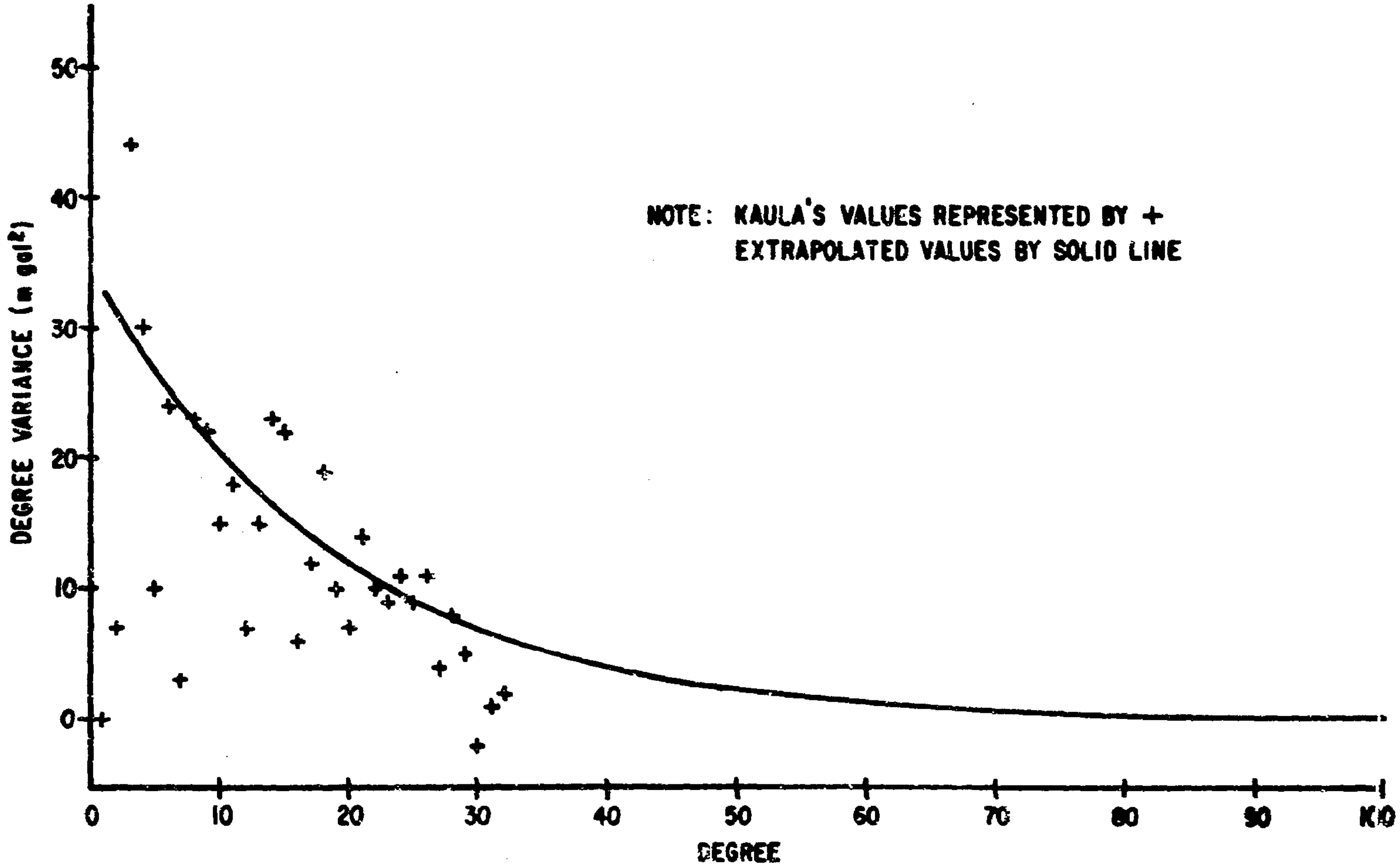

Figure 3. Degree Variances Utilized in Computation

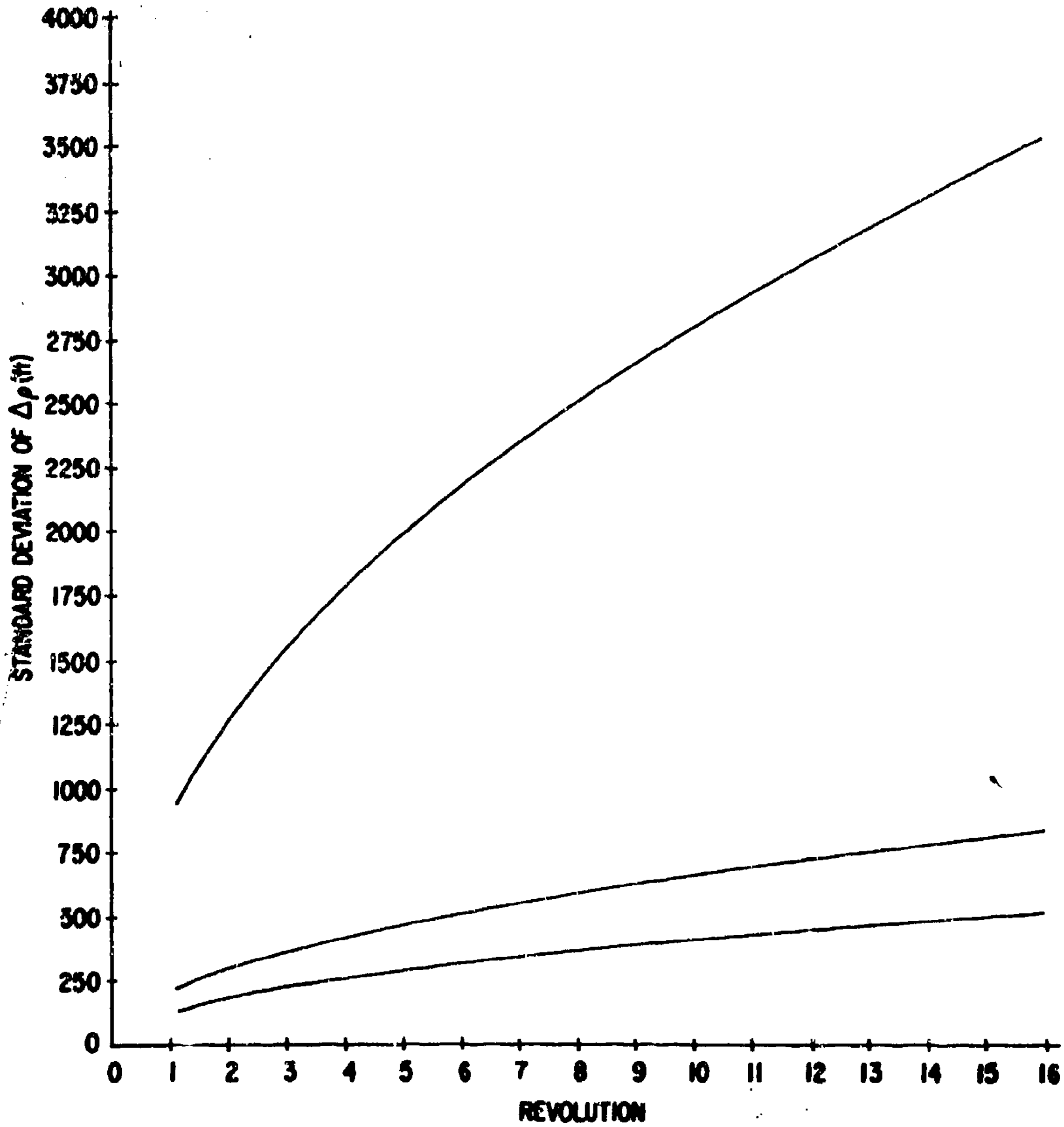

Figure 4. Standard Deviation of $\Delta\rho$ versus Revolution

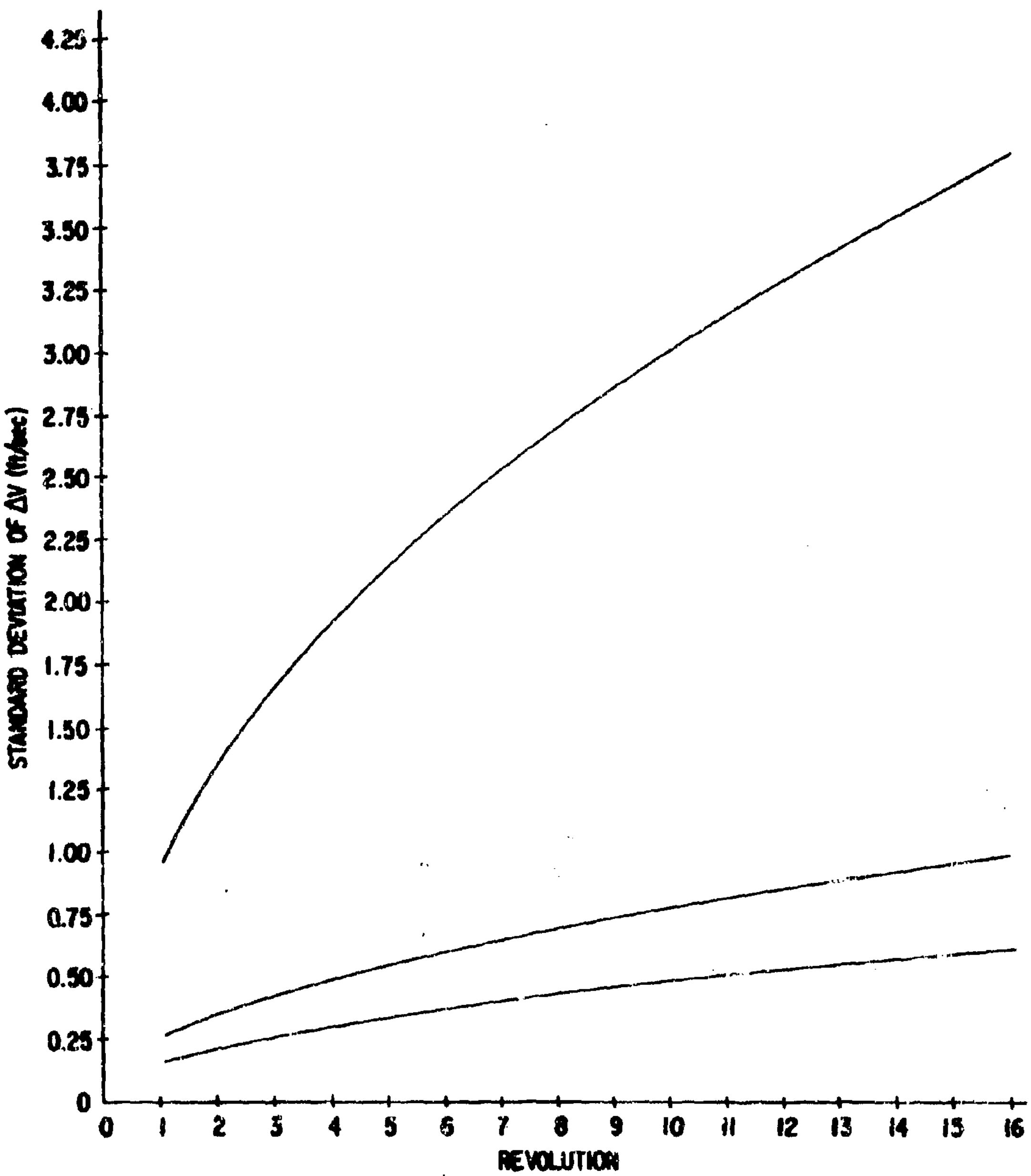

Figure 5. Standard Deviation of Δv versus Revolution

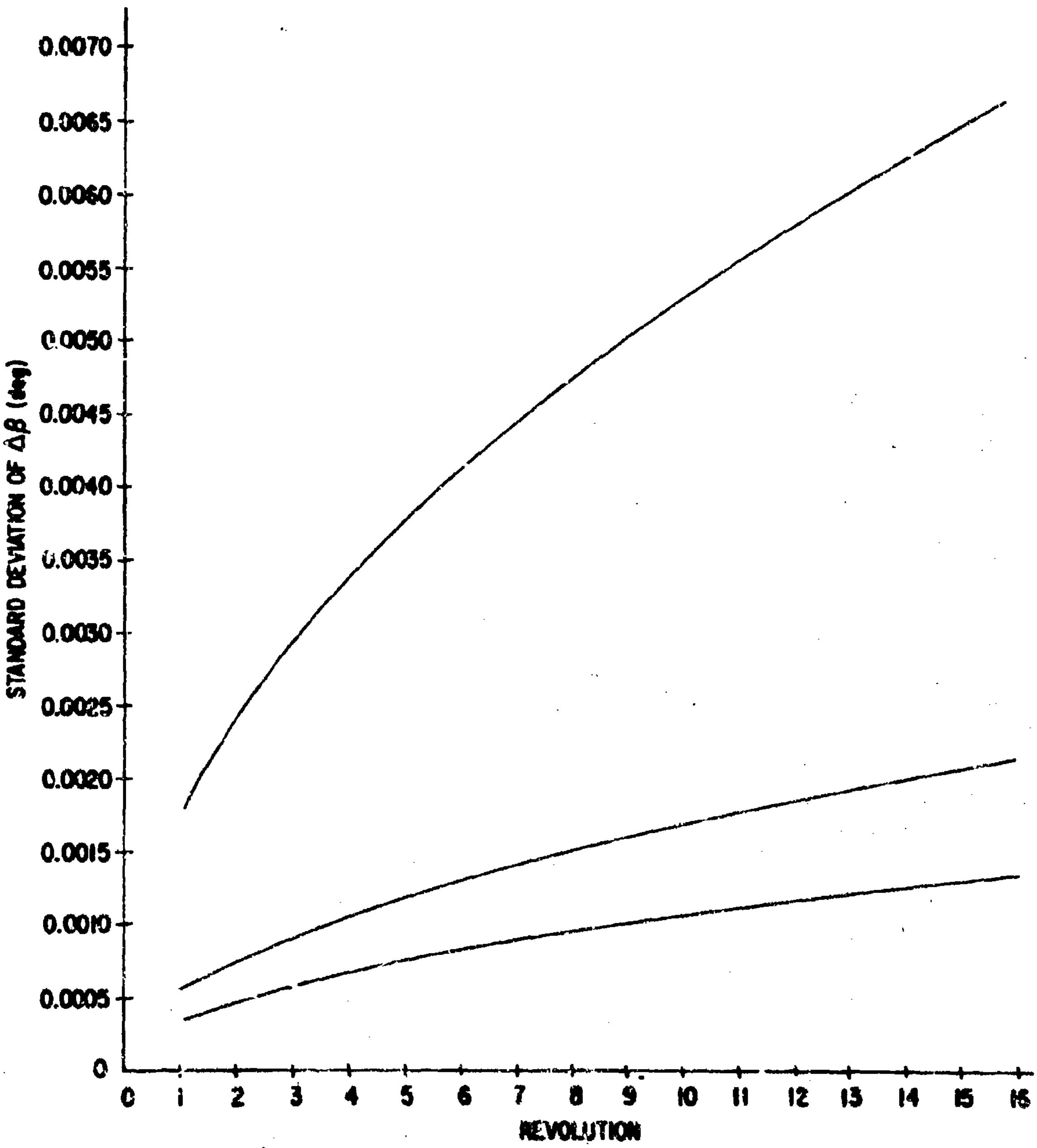

Figure 6. Standard Deviation of Δβ versus Revolution

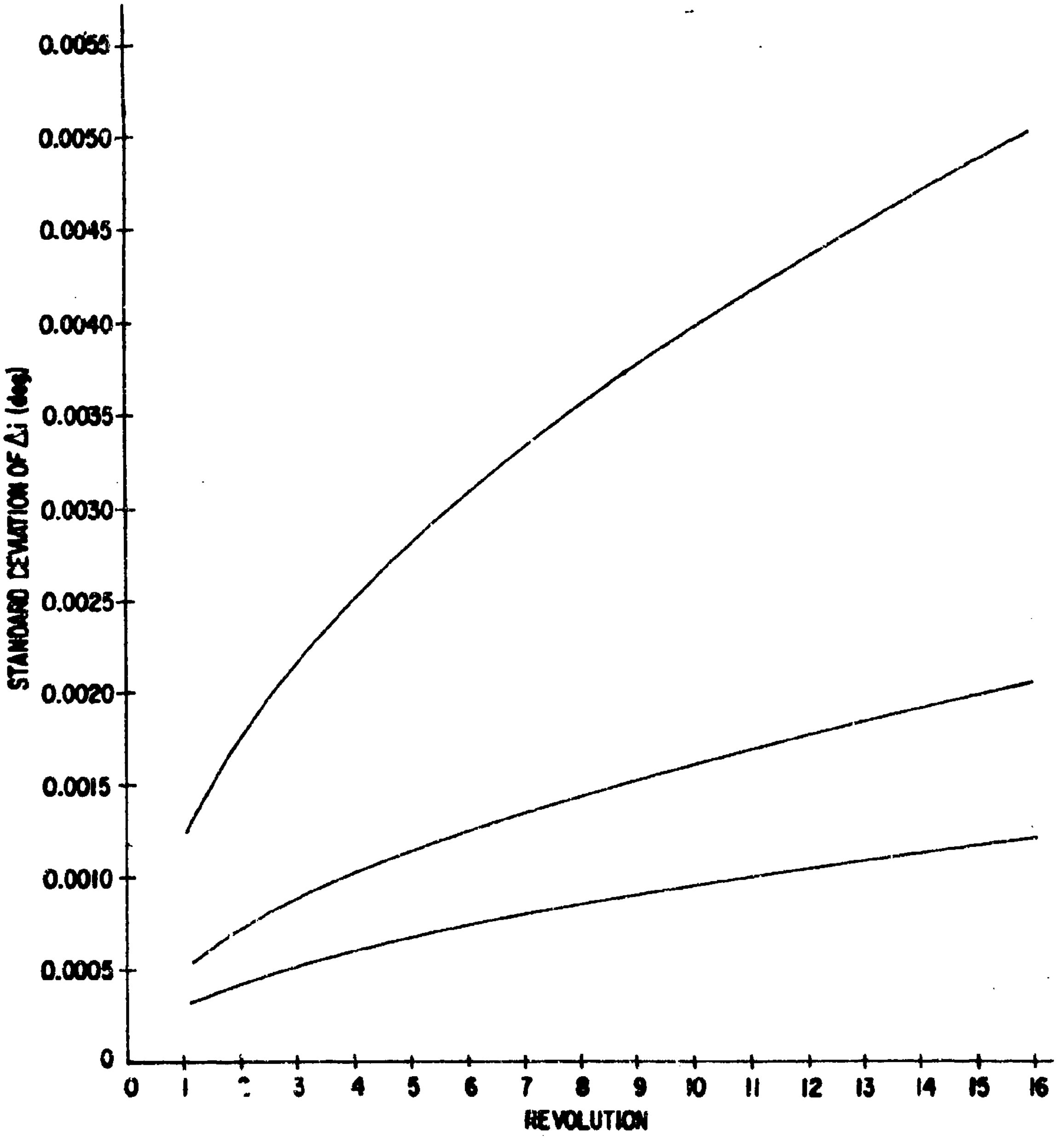

Figure 7. Standard Deviation of Δi versus Revolution

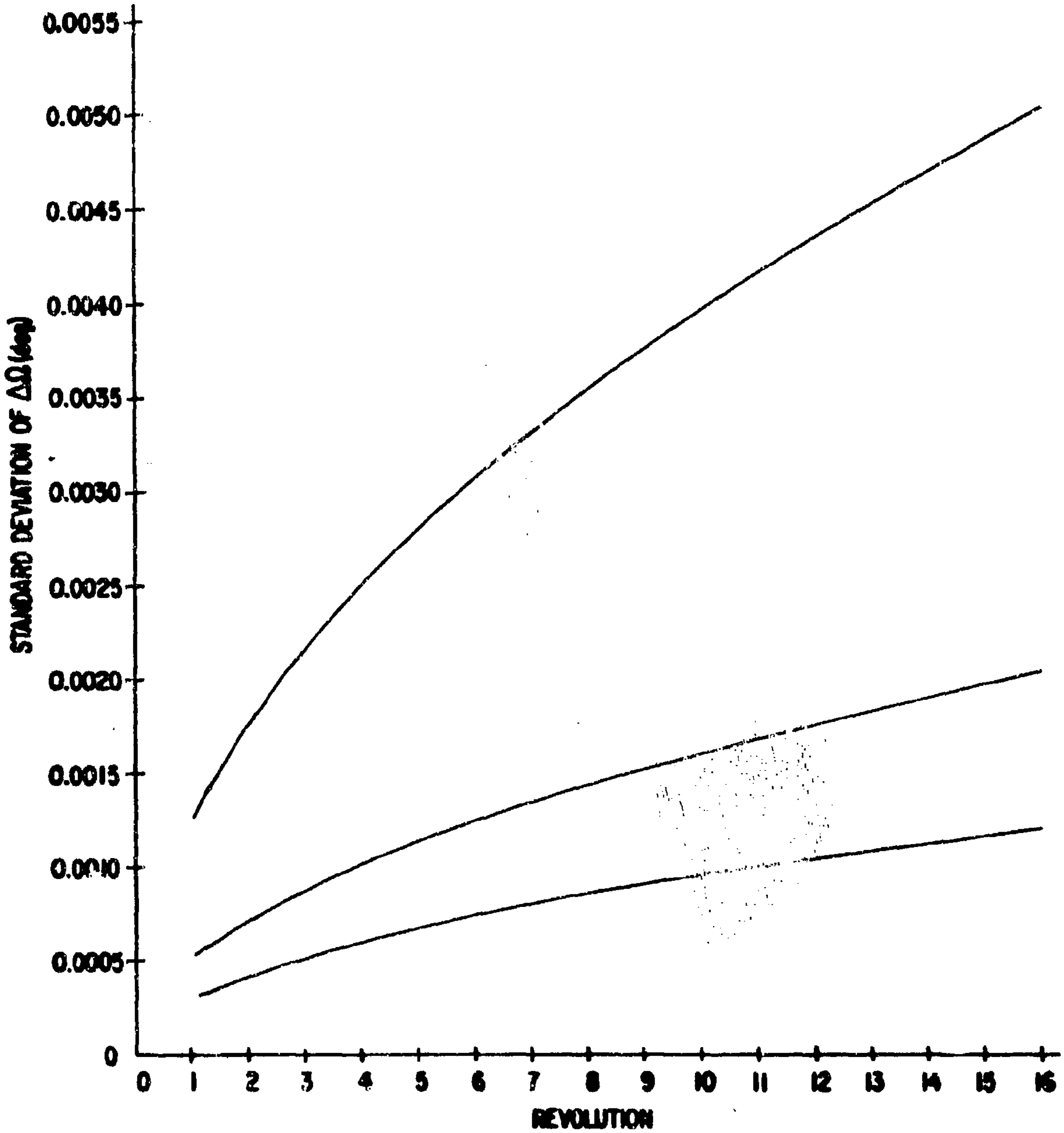

Figure 8. Standard Deviation of $\Delta\Omega$ versus Revolution

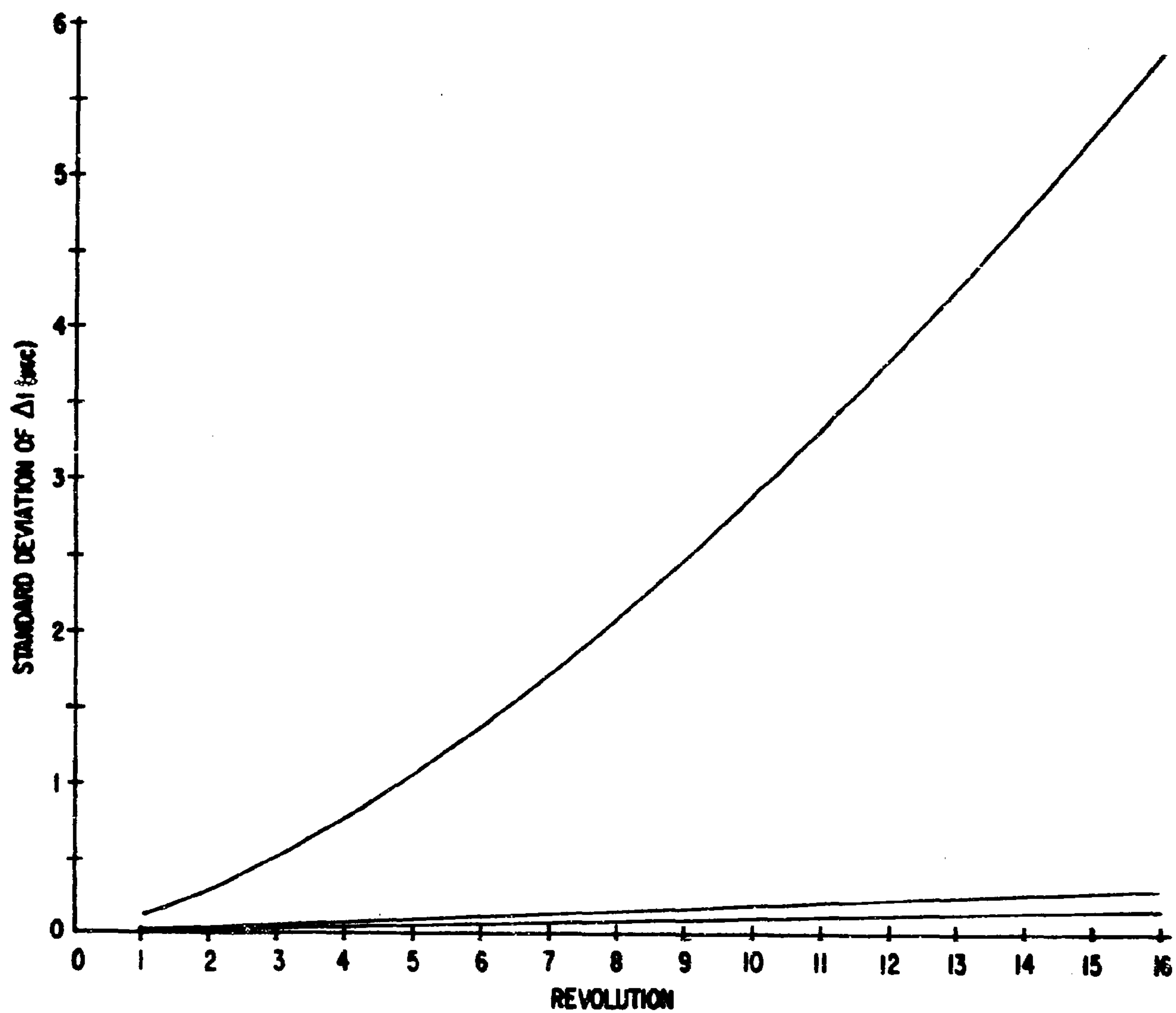

Figure 9. Standard Deviation Δt versus Revolution

effects of these terms on the orbit. Therefore, the curves obtained by summing from n = 7 to 50 correspond to the assumption of a perfectly known gravity model through the 6th degree terms whose effects have been included in the orbit prediction and those obtained by summing from n = 13 to 50 correspond to the assumption of a perfectly known gravity model through the 12th degree terms. Since the degree variances have been calculated from free-air anomalous gravity data, the n = 2 and n = 4 degree variances do not include the preponderant portion of the J_2 and J_4 terms of the terrestrial gravity field. Thus, the n = 2 to 50 case corresponds to an orbit prediction using good approximations to J_2 and J_4 and an exact GM. The decision to set the upper limit of the summation to 50, while somewhat arbitrary, may be justified from examination of Table 1. This table, which was computed assuming a constant degree variance of 10 $mgal^2$ for all terms, presents the contribution to the variance of the orbital elements of selected values of n, normalized with respect to the value associated with n = 2, at the end of 16 satellite revolutions. The fall-off of each of the values with increasing n appears to justify the truncation of the summation at n = 50. Since a satellite in a circular orbit at an altitude of 100 n mi will complete approximately sixteen revolutions in one day, Figs. 4 through 9 yield the standard deviation in the prediction of the orbital elements during an interval of approximately one day. The uncertainties in the orbital elements can be translated into radial, along-track, and cross-track position uncertainties by noting that $\Delta\rho$ is the radial component, Δi and $\Delta\Omega$ can be transformed into the cross-track component for polar orbits, and Δt can be transformed into the along-track component. One-sigma values of these components for a one-day orbital prediction resulting from the use of an incomplete gravity model, as obtained from Figs. 4 through 9, are summarized in Table 2.

Table 1. Variance of Orbital Elements Normalized by Value at n = 2 for Constant Degree Variance

Degree	Variance					
n	a_{11}	a_{22}	a_{33}	a_{44}	a_{55}	a_{66}
2	1.00	1.00	1.00	1.00	1.00	1.00
10	2.84×10^{-3}	8.69×10^{-3}	1.29×10^{-2}	3.46×10^{-2}	3.46×10^{-2}	1.43×10^{-3}
20	8.75×10^{-4}	2.00×10^{-3}	2.98×10^{-3}	8.58×10^{-3}	8.58×10^{-3}	3.43×10^{-4}
30	3.37×10^{-4}	7.05×10^{-4}	1.05×10^{-3}	3.09×10^{-3}	3.09×10^{-3}	1.22×10^{-4}
40	1.45×10^{-4}	2.89×10^{-4}	4.31×10^{-4}	1.33×10^{-3}	1.33×10^{-3}	5.04×10^{-5}
50	6.55×10^{-5}	1.28×10^{-4}	1.91×10^{-4}	5.93×10^{-4}	5.93×10^{-4}	2.23×10^{-5}
100	1.91×10^{-6}	3.53×10^{-6}	5.26×10^{-6}	1.65×10^{-5}	1.65×10^{-5}	6.19×10^{-7}

Table 2. One-Day Orbit Prediction Errors

Gravity Model Utilized in Orbit Prediction	Radial Component (n mi)	Cross-Track Component (n mi)	Along-Track Component (n mi)
GM, J_2, J_4 only	0.58	0.31	24.0
through J_6^6	0.14	0.13	1.1
through J_{12}^{12}	0.085	0.075	0.61

REFERENCES

1. N. S. Hall and H. F. Gawlowicz, "Orbits About an Oblate Pyriform Attracting Body," Proceedings of the Second International Space Science Symposium Florence, Italy, 10-14 April 1961, pp 292-317.

2. D. Brouwer and G. M. Clemence, Methods of Celestial Mechanics, Academic Press, New York and London (1961), pp 273-307.

3. W. M. Kaula, "Statistical and Harmonic Analysis of Gravity," Journal of Geophysical Research, 64 (12), 2401-21 (December 1959).

4. W. M. Kaula, "Determination of the Earth's Gravitational Field," Reviews of Geophysics, 1 (4), 507-50 (November 1963).

5. A. Erdelyi, et al, Higher Transcendental Functions, McGraw-Hill, New York, Toronto, and London (1953), pp 175-82.

APPENDIX A

DERIVATION OF EQUATIONS FOR THE CORRELATION OF THE FORCING FUNCTION

If the ith component of gravity is specified on the surface of a sphere of radius r_o as $g_i(\theta, \lambda)$ where θ and λ are the spherical coordinates, colatitude and longitude, respectively, and the value of the ith component of gravity located at the point θ', λ' with respect to the point θ_o, λ_o in the θ, λ coordinate system is $g_i(\theta', \lambda')$, then the covariance between values of the ith component separated by an angular distance ψ, $C(\psi)$, is

$$C_i(\psi) = \frac{1}{4\pi}\int_{(\theta,\lambda)}\int\frac{1}{2\pi}\int_0^{2\pi} g_i(\theta_o, \lambda_o)g_i(\psi, \lambda')d\lambda' \sin\theta_o d\theta_o d\lambda_o \qquad \text{(A-1)}$$

where the gravity covariance is defined as the average of all possible products of the ith gravity components separated by the arc distance ψ.

The radial component of the covariance is most easily derived and will be considered first. If the gravipotential expansion U is considered to be

$$U(r, \theta, \lambda) = \frac{\mu}{r}\left\{\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n\left[J_nP_n(\cos\theta) + \sum_{m=1}^{n}J_{nm}P_n^m(\cos\theta)\cos m(\lambda - \lambda_{nm})\right]\right\} \qquad \text{(A-2)}$$

where

μ = the product of the universal gravitational constant and the mass of the earth

a = the earth's equatorial radius

r = the vehicular separation from the geocenter

P_n = Legendre polynomials of the first kind of degree n

P_n^m = Legendre associated functions of the first kind of degree n and order m

J_n and J_n^m = coefficients of zonal and non-zonal harmonics, respectively,

then the radial component g_r of anomalous gravity on the sphere is

$$g_r(\theta, \lambda) = \frac{\partial U}{\partial r} = -\frac{\mu}{r_o^2}\left\{\sum_{n=2}^{\infty}(n + 1)\left(\frac{a}{r_o}\right)^n \left[J_n P_n(\cos\theta) + \sum_{m=1}^{n} J_{nm} P_n^m(\cos\theta)\cos m(\lambda - \lambda_{nm})\right]\right\} \tag{A-3}$$

with a similar expansion for $g_r(\theta', \lambda')$

$$g_r(\theta', \lambda') = -\frac{\mu}{r_o^2}\left\{\sum_{n=2}^{\infty}(n + 1)\left(\frac{a}{r_o}\right)^n \left[J_n'(\theta_o, \lambda_o) P_n(\cos\theta) + \sum_{m=1}^{n} J'_{nm}(\theta_o, \lambda_o)\, P_n^m(\cos\theta')\cos m(\lambda' - \lambda'_{nm})\right]\right\} \tag{A-4}$$

where the expression for $J_n'(\theta_o, \lambda_o)$ will be determined later. From Eq. (A-1) the averaging process over the λ' coordinate is only performed on $g_r(\theta', \lambda')$. Hence, all non-zonal terms will average to 0. Taking the summation outside the integral yields for (A-1)

$$C_r(\psi) = -\frac{\mu}{r_o^2}\sum_{n=2}^{\infty}\frac{(n + 1)}{4\pi}\left(\frac{a}{r_o}\right)^n P_n(\cos\psi)\int_o^{\pi}\int_o^{2\pi} g_r(\theta_o, \lambda_o) J_n'(\theta_o, \lambda_o)\sin\theta_o\, d\theta_o\, d\lambda_o \tag{A-5}$$

The relationship for J_n' can be established from the following considerations. At every point on the sphere $g_r(\theta', \lambda') = g_r(\theta, \lambda)$; therefore

$$\frac{1}{S'}\iint_{S'} P_n(\cos \theta')\, g_r(\theta', \lambda')\, dS' = \frac{1}{S}\iint_{S} P_n(\cos \theta')\, g_r(\theta, \lambda)\, dS \tag{A-6}$$

where S is the spherical surface and dS is an element of area. From the orthogonality properties of $P_n(\cos \theta)$, the left side of (A-6) gives

$$\frac{J_n'(\theta_o, \lambda_o)}{2n + 1}$$

Now

$$P_n(\cos \theta') = P_n(\cos \theta)\, P_n(\cos \theta_o) + 2 \sum_{m=1}^{n} \frac{(n - m)!}{(n + m)!}$$

$$\cdot P_n^m(\cos \theta)\, P_n^m(\cos \theta_o) \cos[m(\lambda - \lambda_o)] \tag{A-7}$$

If we let $J_{nm} \cos m\lambda_{nm} = A_{nm}$, $J_{nm} \sin m\lambda_{nm} = B_{nm}$, then integration of the right side of (A-6) yields

$$J_n'(\theta_o, \lambda_o) = J_n P_n(\cos \theta_o) + \sum_{m=1}^{n} [A_{nm} P_n^m(\cos \theta_o) \cos m\lambda_o$$

$$+ B_{nm} P_n^m(\cos \theta_o) \sin m\lambda_o] \tag{A-8}$$

Define the normalization factor for spherical harmonics as

$$N^2(n, m) = \frac{1}{(2n + 1)(2 - \delta_{mo})} \frac{(n + m)!}{(n - m)!} \tag{A-9}$$

where $\delta_{mo} = \begin{cases} 1 \text{ if } m=0 \\ 0 \text{ if } m\neq 0 \end{cases}$ is the Kronecker delta.

Then, substituting (A-8) into (A-5) allows the integration to be performed by inspection yielding

$$C_r(\psi) = \left(\frac{\mu}{r_o^2}\right)^2 \sum_{n=2}^{\infty} (n+1)^2 \left(\frac{a}{r_o}\right)^{2n} P_n(\cos\psi) \cdot \left\{ N^2(n,0)\, J_n^2 + \sum_{m=1}^{n} N^2(n,m)\left(A_{nm}^2 + B_{nm}^2\right)\right\}$$

(A-10)

or using the bar superscript to indicate a normalized coefficient

$$C_r(\psi) = \left(\frac{\mu}{r_o^2}\right)^2 \sum_{n=2}^{\infty} (n+1)^2 \left(\frac{a}{r_o}\right)^{2n} P_n(\cos\psi) \cdot \left\{ \bar{J}_n^2 + \sum_{m=1}^{n}\left(\bar{A}_{nm}^2 + \bar{B}_{nm}^2\right)\right\} \quad \text{(A-11)}$$

Kaula (Ref. 3) defines the anomalous gravity covariance on the earth's surface by

$$C(S) = \sum_{n=2}^{\infty} \sigma_n^2 P_n(\cos S) \qquad \text{(A-12)}$$

where the degree variance

$$\sigma_n^2 = \sum_{m=0}^{n}\left(\bar{\alpha}_{nm}^2 + \bar{\beta}_{nm}^2\right) \qquad \text{(A-13)}$$

Recalling that

$$\begin{Bmatrix} \bar{\alpha}_{nm} \\ \bar{\beta}_{nm} \end{Bmatrix} \frac{a^2}{\mu(n-1)} = \begin{Bmatrix} \bar{A}_{nm} \\ \bar{B}_{rm} \end{Bmatrix}$$

Eq. (A-11) may be expressed in terms of Kaula's degree variances by

$$C_r(\psi) = \left(\frac{a}{r_o}\right)^4 \sum_{n=2}^{\infty} \left(\frac{n+1}{n-1}\right)^2 \left(\frac{a}{r_o}\right)^{2n} \sigma_n^2 \, P_n(\cos\psi) \qquad \text{(A-14)}$$

The covariance of the horizontal components of the gravity is considerably more difficult to derive by the methods used to obtain the covariance of the radial component. However, Kaula (Ref. 3) gives the covariance of the deflection of the vertical in terms of the component ρ parallel to the great circle arc joining the two points and the component τ perpendicular to the line joining the two points. These components, in the case of interest here, represent the along-track and the cross-track component, respectively. Noting that, to a very good approximation, the horizontal component of the anomalous gravity field is just the total gravity force multiplied by the deflection angles (in radians), Kaula's results can be used to obtain for the along-track covariance at the earth's surface

$$C_s(\psi) = \sum_{n=2}^{\infty} \frac{(n+1)n}{2(n-1)^2} \sigma_n^2 \left[P_n(\cos\psi) - \frac{P_n^2(\cos\psi)}{n(n+1)}\right] \qquad \text{(A-15)}$$

and for the cross-track covariance

$$C_w(\psi) = \sum_{n=2}^{\infty} \frac{(n+1)n}{2(n-1)^2} \sigma_n^2 \left[P_{n-1}(\cos\psi) + \frac{P_{n-1}^2(\cos\psi)}{n(n+1)}\right] \qquad \text{(A-16)}$$

where $P_n^2(\cos\psi)$ and $P_{n-1}^2(\cos\psi)$ are Legendre associated functions. Kaula obtained the above results in terms of σ_n^2 by differentiating the expression relating geoid height to the geopotential and squaring. On a sphere of radius r_o, the expression for the radial covariance of degree n is given by the product of $\left(\frac{a}{r_o}\right)^{2n+4}$ and the expression on the surface. Hence, to convert

Kaula's results to a sphere of arbitrary radius r_o, each term of the summation of the right hand sides of Eqs. (A-15) and (A-16) must be multiplied by $\left(\frac{a}{r_o}\right)^{2n+4}$ yielding

$$C_s(\psi) = \left(\frac{a}{r_o}\right)^4 \sum_{n=2}^{\infty} \frac{(n+1)n}{2(n-1)^2} \left(\frac{a}{r_o}\right)^{2n} \sigma_n^2 \left[P_n(\cos\psi) - \frac{P_n^2(\cos\psi)}{n(n+1)}\right] \quad \text{(A-17)}$$

$$C_w(\psi) = \left(\frac{a}{r_o}\right)^4 \sum_{n=2}^{\infty} \frac{(n+1)n}{2(n-1)^2} \left(\frac{a}{r_o}\right)^{2n} \sigma_n^2 \left[P_{n-1}(\cos\psi) + \frac{P_{n-1}^2(\cos\psi)}{n(n+1)}\right] \quad \text{(A-18)}$$

However, it is noted (see e.g., Ref. 5) that

$$P_n(\cos\psi) = \sum_{k=0}^{n} 2^{-2n} \binom{2k}{k} \binom{2n-2k}{n-k} \cos(n-2k)\psi \quad \text{(A-19)}$$

and that

$$P_n^2(\cos\psi) = \frac{3}{2} \sum_{k=0}^{n-2} \frac{\Gamma\left(k+\frac{5}{2}\right)\Gamma\left(n-k+\frac{1}{2}\right)}{\Gamma^2\left(\frac{5}{2}\right) k!\,(n-k-2)!}$$

$$\cdot\left\{\cos(n-2k-2)\psi - \frac{1}{2}\cos(n-2k-4)\psi - \frac{1}{2}\cos(n-2k)\psi\right\} \quad \text{(A-20)}$$

Thus, one sees that all the covariance functions can be expressed as linear combinations of cos $(\lambda\psi)$ terms where λ is an arbitrary integer. In other words, substitution of Eqs. (A-19) and (A-20) into Eqs. (A-14), (A-17), and (A-18) yields covariance components of the form given by Eq. (6-f) in the text.

APPENDIX B

EXPRESSIONS FOR ELEMENTS APPEARING IN THE COVARIANCE MATRIX OF THE ORBITAL ELEMENTS

Lower bounds for the covariance of the orbital elements are obtained from direct integration of Eq. (7) where the elements of the integrand are given by Eqs. (6-d), (6-e), and (6-f). The resultant solution is given by the symmetrical matrix, Eq. (8), in the text.

$$\mathrm{COV}\begin{bmatrix}\Delta\rho\\ \Delta v\\ \Delta\beta\\ \Delta i\\ \Delta\Omega\\ \Delta t\end{bmatrix} = \begin{bmatrix} a_{11}, & a_{12}, & a_{13}, & 0, & 0, & a_{16}\\ a_{21}, & a_{22}, & a_{23}, & 0, & 0, & a_{26}\\ a_{31}, & a_{32}, & a_{33}, & 0, & 0, & a_{36}\\ 0, & 0, & 0, & a_{44}, & a_{45}, & a_{46}\\ 0, & 0, & 0, & a_{54}, & a_{55}, & a_{56}\\ a_{61}, & a_{62}, & a_{63}, & a_{64}, & a_{65}, & a_{66}\end{bmatrix} \qquad \text{(B-1)}$$

where

$$a_{11} = \frac{\rho_o^4}{v_o^4}(\alpha X_{11} + 4\beta Y_{11})$$

$$a_{12} = a_{21} = \frac{\rho_o^3}{v_o^3}(-\alpha X_{12} + 2\beta Y_{12})$$

$$a_{13} = a_{31} = -\frac{\rho_o^3}{v_o^4}(\alpha X_{13} + 4\beta Y_{13})$$

$$a_{16} = a_{61} = 2\frac{\rho_o^4}{v_o^5}(\alpha X_{16} + \beta Y_{16})$$

$$a_{22} = \frac{\rho_o^2}{v_o^2}(\alpha X_{22} + \beta Y_{22})$$

$$a_{23} = a_{32} = \frac{\rho_o^2}{v_o^3}(\alpha X_{23} - 2\beta Y_{23})$$

$$a_{26} = a_{62} = \frac{\rho_o^3}{v_o^4}(-2\alpha X_{26} + \beta Y_{26})$$

$$a_{33} = \frac{\rho_o^2}{v_o^4}(\alpha X_{33} + 4\beta Y_{33})$$

$$a_{36} = a_{63} = -2\frac{\rho_o^3}{v_o^5}(\alpha X_{36} + \beta Y_{36})$$

$$a_{44} = \frac{\rho_o^2}{v_o^4}\gamma Z_{44}$$

$$a_{45} = a_{54} = -\frac{\rho_o^2}{v_o^4}\frac{1}{\sin i}\gamma Z_{45}$$

$$a_{46} = a_{64} = -\frac{\rho_o^3}{v_o^5}\frac{\cos i}{\sin i}\gamma Z_{46}$$

$$a_{55} = \frac{\rho_o^2}{v_o^4} \frac{1}{\sin^2 i} \gamma Z_{55}$$

$$a_{56} = a_{65} = \frac{\rho_o^3}{v_o^5} \frac{\cos i}{\sin^2 i} \gamma Z_{56}$$

$$a_{66} = \frac{\rho_o^4}{v_o^6} \left(\alpha X_{66} + \beta Y_{66} + \frac{\cos^2 i}{\sin^2 i} \gamma Z_{66} \right)$$

and

$$\begin{aligned}
X_{11} = \frac{1}{2(\lambda^2 - 1)^2} \Big[& 2 \big\{ \cos \lambda A [(\zeta - \zeta_o - A) \sin A - \cos A] \\
& + \cos (\zeta - \zeta_o)[\cos(\zeta - \zeta_o) - \cos \lambda A \cos(\zeta - \zeta_o - A)] + 1 \big\} \\
& + \lambda \sin \lambda A \big\{ \cos A [-2(\zeta - \zeta_o - A) + \sin 2(\zeta - \zeta_o - A)] \\
& \quad - 5 \sin A - \cos(\zeta - \zeta_o - 2A) \sin(\zeta - \zeta_o - A) \\
& \quad + \sin(\zeta - \zeta_o) \cos(\zeta - \zeta_o - A) \big\} \\
& + 2\lambda^2 \big\{ -(\zeta - \zeta_o - A) \sin A \cos \lambda A \\
& \quad + \sin (\zeta - \zeta_o)[\sin(\zeta - \zeta_o) - \cos \lambda A \sin(\zeta - \zeta_o - A)] \big\} \\
& + \lambda^3 \sin \lambda A \big\{ \cos A [2(\zeta - \zeta_o - A) - \sin 2(\zeta - \zeta_o - A)] + \sin A \\
& \quad - \sin(\zeta - \zeta_o) \cos(\zeta - \zeta_o - A) \\
& \quad + \cos(\zeta - \zeta_o - 2A) \sin(\zeta - \zeta_o - A) \big\} \Big]
\end{aligned}$$

$$
\begin{aligned}
Y_{11} = \frac{1}{2\lambda^2(\lambda^2 - 1)^2} \Big[& 4\,(1 - \cos\lambda A) + 4\lambda \sin\lambda A \,[(\zeta - \zeta_o - A) - \sin(\zeta - \zeta_o)] \\
& + 2\lambda^2 \big\{ \cos\lambda A [\,(\zeta - \zeta_o - A)\sin A - \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A) \\
& \qquad + 4 - 2\cos(\zeta - \zeta_o - A) - 2\cos A\,] \\
& \quad + \sin^2(\zeta - \zeta_o) - 2 + 2\cos(\zeta - \zeta_o) \big\} \\
& + \lambda^3 \sin\lambda A \,[\,-2(\zeta - \zeta_o - A)(4 + \cos A) - 7\sin A \\
& \qquad + \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A) \\
& \qquad - \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A) + 4\sin(\zeta - \zeta_o - A) \\
& \qquad + 8\sin(\zeta - \zeta_o) - \cos A \sin 2(\zeta - \zeta_o - A)\,] \\
& + 2\lambda^4 \big\{ \cos\lambda A [\,-(\zeta - \zeta_o - A)\sin A - 2 + 2\cos(\zeta - \zeta_o - A) \\
& \qquad + \cos A - \cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\,] \\
& \quad + 1 + \cos^2(\zeta - \zeta_o) - 2\cos(\zeta - \zeta_o) \big\} \\
& + \lambda^5 \sin\lambda A \,[\,2(\zeta - \zeta_o - A)(2 + \cos A) - 4\sin(\zeta - \zeta_o - A) \\
& \qquad - 4\sin(\zeta - \zeta_o) + 3\sin A + \cos A \sin 2(\zeta - \zeta_o - A) \\
& \qquad - \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A) \\
& \qquad + \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\,] \Big]
\end{aligned}
$$

$$X_{12} = X_{11}$$

$$\begin{aligned}
Y_{12} = \frac{1}{\lambda^2(\lambda^2 - 1)^2}\Big[& -2(1 - \cos\lambda A) + \lambda \sin\lambda A[-2(\zeta - \zeta_o - A) + 3\sin(\zeta - \zeta_o)] \\
& + \lambda^2\{\cos\lambda A[-2(\zeta - \zeta_o - A)\sin A + 3\cos(\zeta - \zeta_o - A) \\
& \qquad + 3\cos A - 4 + 2\sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)] \\
& \quad - 3\cos(\zeta - \zeta_o) + 1 - 2\sin^2(\zeta - \zeta_o)\} \\
& + \lambda^3 \sin\lambda A[2(\zeta - \zeta_o - A)(2 + \cos A) - 6\sin(\zeta - \zeta_o) \\
& \qquad + 6\sin A - 3\sin(\zeta - \zeta_o - A) \\
& + \cos A \sin 2(\zeta - \zeta_o - A) - \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A) \\
& + \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)] \\
& + \lambda^4\{\cos\lambda A[2(\zeta - \zeta_o - A)\sin A - 3\cos(\zeta - \zeta_o - A) \\
& \qquad - \cos A + 2 + 2\cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)] \\
& \quad + 3\cos(\zeta - \zeta_o) - 1 - 2\cos^2(\zeta - \zeta_o)\} \\
& + \lambda^5 \sin\lambda A[-2(\zeta - \zeta_o - A)(1 + \cos A) + 3\sin(\zeta - \zeta_o) \\
& \qquad + 3\sin(\zeta - \zeta_o - A) - 2\sin A \\
& \qquad - \cos A \sin 2(\zeta - \zeta_o - A) \\
& \qquad + \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A) \\
& \qquad - \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)]\Big]
\end{aligned}$$

$$X_{13} = \frac{1}{2(\lambda^2 - 1)^2} \Big\{ \cos \lambda A [3 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- \sin A \cos 2(\zeta - \zeta_o - A)$$

$$- \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)] - \sin 2(\zeta - \zeta_o)$$

$$- 2\lambda \sin \lambda A \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)$$

$$+ \lambda^2 \{\cos \lambda A [\sin A \cos 2(\zeta - \zeta_o - A) - 2 \sin A$$

$$+ \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$- \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- 2 \cos(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)] + \sin 2(\zeta - \zeta_o)\}$$

$$+ 2\lambda^3 \sin \lambda A \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)\Big]$$

$$Y_{13} = \frac{1}{2\lambda(\lambda^2 - 1)^2} \Big[2 \sin \lambda A [1 - \cos(\zeta - \zeta_o)]$$

$$+ \lambda \{\cos \lambda A [2 \sin (\zeta - \zeta_o - A)$$

$$+ 2 \sin A - 3 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$+ \sin A \cos 2(\zeta - \zeta_o - A)$$

$$+ \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)]$$

$$- 2 \sin(\zeta - \zeta_o)[1 - \cos(\zeta - \zeta_o)]\}$$

$$+ 2\lambda^2 \sin \lambda A [- 2 + 2 \cos(\zeta - \zeta_o) - \cos A + \cos(\zeta - \zeta_o - A)$$

$$+ \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$+ \lambda^3 \{\cos \lambda A [-2 \sin(\zeta - \zeta_o - A) - \sin A \cos 2(\zeta - \zeta_o - A)$$

$$- \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$+ 2\cos(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$+ 2\sin(\zeta - \zeta_o)[1 - \cos(\zeta - \zeta_o)]\}$$

$$+ 2\lambda^4 \sin \lambda A[1 + \cos A - \cos(\zeta - \zeta_o) - \cos(\zeta - \zeta_o - A)$$

$$- \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]\Big]$$

$$X_{16} = Y_{13}$$

$$Y_{16} = \frac{1}{\lambda^2(\lambda^2 - 1)^2}\Big[3[(\zeta - \zeta_o) - \sin(\zeta - \zeta_o)](1 - \cos \lambda A)$$

$$+ \lambda \sin \lambda A \{3(\zeta - \zeta_o)^2 - 3(\zeta - \zeta_o)[A + \sin(\zeta - \zeta_o)]$$

$$+ 3A\sin(\zeta - \zeta_o) - 1 + \cos(\zeta - \zeta_o)\}$$

$$+ \lambda^2\{3(\zeta - \zeta_o)[-2 + 2\cos \lambda A + \cos(\zeta - \zeta_o)$$

$$- \cos \lambda A \cos(\zeta - \zeta_o - A)]$$

$$+ \cos \lambda A[-6\sin(\zeta - \zeta_o) - \sin(\zeta - \zeta_o - A) - \sin A$$

$$+ 6\sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- 2\sin A \cos 2(\zeta - \zeta_o - A)$$

$$- \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ \sin(\zeta - \zeta_o)[7 - 4\cos(\zeta - \zeta_o)]\}$$

$$+ \lambda^3 \sin \lambda A\{- 6(\zeta - \zeta_o)^2 + 3(\zeta - \zeta_o)[2A + 2\sin(\zeta - \zeta_o)$$

$$+ \sin(\zeta - \zeta_o - A)]$$

$$- 6A\sin(\zeta - \zeta_o) + 2 - 2\cos(\zeta - \zeta_o)$$

$$+ \cos A - \cos(\zeta - \zeta_o - A)$$

$$- 4\sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)\}$$

$$+ \lambda^4 \{3(\zeta - \zeta_o)[1 - \cos\lambda A - \cos(\zeta - \zeta_o)$$

$$+ \cos\lambda A\cos(\zeta - \zeta_o - A)]$$

$$+ \cos\lambda A[- 3\sin A + 3\sin(\zeta - \zeta_o) + \sin(\zeta - \zeta_o - A)$$

$$+ 2\sin A\cos 2(\zeta - \zeta_o - A)$$

$$+ 2\sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$- 2\sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- 4\cos(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$-4\sin(\xi - \zeta_o)[1 - \cos(\zeta - \zeta_o)]\}$$

$$+ \lambda^5 \sin\lambda A\{3(\zeta - \zeta_o)^2 - 3(\zeta - \zeta_o)[A + \sin(\zeta - \zeta_o)$$

$$+ \sin(\zeta - \zeta_o - A)]$$

$$+ 3A\sin(\zeta - \zeta_o) - 1 - \cos A + \cos(\zeta - \zeta_o)$$

$$+ \cos(\zeta - \zeta_o - A) + 4\sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)\}]$$

$$X_{22} = X_{11}$$

$$
\begin{aligned}
Y_{22} = \frac{2}{\lambda^2(\lambda^2 - 1)^2} \Big[& 1 - \cos\lambda A + \lambda \sin\lambda A [(\zeta - \zeta_o - A) - 2\sin(\zeta - \zeta_o)] \\
& + 2\lambda^2 \{\cos\lambda A [(\zeta - \zeta_o - A)\sin A \\
& \qquad - \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A) - \cos(\zeta - \zeta_o - A) \\
& \qquad - \cos A + 1] \\
& \qquad + \cos(\zeta - \zeta_o) + \sin^2(\zeta - \zeta_o)\} \\
& + \lambda^3 \sin\lambda A [- 2(\zeta - \zeta_o - A)(1 + \cos A) \\
& \qquad - \sin 2(\zeta - \zeta_o - A)\cos A + 2\sin(\zeta - \zeta_o - A) \\
& \qquad + 4\sin(\zeta - \zeta_o) - 5\sin A \\
& \qquad + \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A) \\
& \qquad - \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)] \\
& + \lambda^4 \{\cos\lambda A [- 2(\zeta - \zeta_o - A)\sin A \\
& \qquad + 2\cos(\zeta - \zeta_o - A) \\
& \qquad - 2\cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A) - 1] \\
& \qquad - 2\cos(\zeta - \zeta_o)[1 - \cos(\zeta - \zeta_o)] + 1\}
\end{aligned}
$$

$$+ \lambda^5 \sin \lambda A \left[(\zeta - \zeta_o - A)(1 + 2 \cos A) \right.$$

$$+ \sin 2(\zeta - \zeta_o - A) \cos A$$

$$- \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A) - 2 \sin(\zeta - \zeta_o)$$

$$\left. - 2 \sin(\zeta - \zeta_o - A) + \sin A \right] \Big]$$

$$X_{23} = X_{13}$$

$$Y_{23} = \frac{1}{\lambda(\lambda^2 - 1)^2} \Big[- \sin \lambda A \left[1 - \cos (\zeta - \zeta_o) \right]$$

$$+ \lambda \Big\{ \cos \lambda A \left[3 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A) \right.$$

$$- \sin A \cos 2 (\zeta - \zeta_o - A)$$

$$- \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$\left. - \sin(\zeta - \zeta_o - A) - \sin A \right] + \sin(\zeta - \zeta_o)$$

$$- \sin 2(\zeta - \zeta_o) \Big\}$$

$$+ \lambda^2 \sin \lambda A \left[2 + \cos A - 2 \cos(\zeta - \zeta_o) - \cos(\zeta - \zeta_o - A) \right.$$

$$\left. - 2 \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A) \right]$$

$$+ \lambda^3 \Big\{ \cos \lambda A \left[\sin A \cos 2(\zeta - \zeta_o - A) - \sin A \right.$$

$$+ \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$- \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- 2 \cos(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)$$

$$+ \sin(\zeta - \zeta_o - A)] - \sin(\zeta - \zeta_o)$$

$$+ \sin 2(\zeta - \zeta_o)\}$$

$$+ \lambda^4 \sin \lambda A \left[- 1 - \cos A + \cos(\zeta - \zeta_o) + \cos(\zeta - \zeta_o - A) \right.$$

$$\left. + 2 \sin (\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)] \right]$$

$$X_{26} = Y_{13}$$

$$Y_{26} = \frac{1}{\lambda^2(\lambda^2 - 1)^2} \left[- 3[(\zeta - \zeta_o) - 2 \sin (\zeta - \zeta_o)](1 - \cos \lambda A) \right.$$

$$+ \lambda \sin \lambda A\{-3(\zeta - \zeta_o)^2 + 3(\zeta - \zeta_o)[2 \sin(\zeta - \zeta_o) + A]$$

$$- 6A \sin(\zeta - \zeta_o) + 2 \cos(\zeta - \zeta_o) - 2\}$$

$$+ 2\lambda^2 \{3(\zeta - \zeta_o)[- \cos(\zeta - \zeta_o) + \cos \lambda A \cos(\zeta - \zeta_o - A)$$

$$+ 1 - \cos \lambda A]$$

$$+ \cos \lambda A [- \sin A - \sin(\zeta - \zeta_o - A) + 6 \sin(\zeta - \zeta_o)$$

$$- 6 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$+ 2 \sin A \cos 2(\zeta - \zeta_o - A)$$

$$+ 2 \sin (\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)]$$

$$- 5 \sin (\zeta - \zeta_o) + 2 \sin 2(\zeta - \zeta_o)\}$$

$$+ 2\lambda^3 \sin \lambda A\{3(\zeta - \zeta_o)^2 - 3(\zeta - \zeta_o)[A + 2 \sin (\zeta - \zeta_o)$$

$$+ \sin (\zeta - \zeta_o - A)]$$

$$+ 6A \sin(\zeta - \zeta_o) - 2 \cos(\zeta - \zeta_o)$$

$$- \cos(\zeta - \zeta_o - A) + \cos A + 2$$

$$+ 4 \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)\}$$

$$+ \lambda^4 \{3(\zeta - \zeta_o)[2 \cos(\zeta - \zeta_o) - 2 \cos \lambda A \cos(\zeta - \zeta_o - A) - 1$$

$$+ \cos \lambda A]$$

$$+ 2 \cos \lambda A[5 \sin A + \sin(\zeta - \zeta_o - A) - 3 \sin(\zeta - \zeta_o)$$

$$- 2 \sin A \cos 2(\zeta - \zeta_o - A)$$

$$- 2 \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ 2 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$+ 4 \cos(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$+ 4 \sin(\zeta - \zeta_o) - 4 \sin 2(\zeta - \zeta_o)\}$$

$$+ \lambda^5 \sin \lambda A\{- 3(\zeta - \zeta_o)^2 + 3(\zeta - \zeta_o)[2 \sin(\zeta - \zeta_o)$$

$$+ 2 \sin(\zeta - \zeta_o - A) + A]$$

$$- 6A \sin(\zeta - \zeta_o) + 2 \cos(\zeta - \zeta_o)$$

$$+ 2 \cos(\zeta - \zeta_o - A) - 2 \cos A - 2$$

$$- 8 \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o A)\}]$$

$$X_{33} = \frac{1}{2(\lambda^2 - 1)^2} \Big[2[(\zeta - \zeta_o - A) \sin A \cos \lambda A + \sin^2(\zeta - \zeta_o)$$

$$- \cos \lambda A \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$+ \lambda \sin \lambda A \{\cos A [-2(\zeta - \zeta_o - A) - \sin 2(\zeta - \zeta_o - A)]$$

$$- 3 \sin A + \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$- \sin (\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\}$$

$$+ 2\lambda^2 [- (\zeta - \zeta_o - A)\sin A \cos \lambda A - \cos A \cos \lambda A + 1$$

$$+ \cos^2(\zeta - \zeta_o) - \cos \lambda A \cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)]$$

$$+ \lambda^3 \sin \lambda A \{\cos A[2(\zeta - \zeta_o - A) + \sin 2(\zeta - \zeta_o - A)]$$

$$- \sin A - \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ \sin (\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\}\Big]$$

$$Y_{33} = X_{11}$$

$$X_{36} = \frac{1}{2\lambda (\lambda^2 - 1)^2} \Big[2 \sin \lambda A \sin (\zeta - \zeta_o)$$

$$+ 2\lambda \{ \cos \lambda A [-(\zeta - \zeta_o - A) \sin A + \sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)$$

$$+ \cos(\zeta - \zeta_o - A) + \cos A]$$

$$- 1 - \cos(\zeta - \zeta_o) - \sin^2(\zeta - \zeta_o)\}$$

$$+ \lambda^2 \sin \lambda A \{\cos A [2(\zeta - \zeta_o - A) + \sin 2(\zeta - \zeta_o - A)]$$

$$- 2 \sin(\zeta - \zeta_o - A) - 4 \sin (\zeta - \zeta_o)$$

$$+ 5 \sin A - \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$+ \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\}$$

$$+ 2\lambda^3 \{\cos \lambda A[(\zeta - \zeta_o - A)\sin A + \cos(\zeta - \zeta_o)(\cos(\zeta - \zeta_o - A)$$

$$- \cos(\zeta - \zeta_o - A)]$$

$$+ \cos(\zeta - \zeta_o)[1 - \cos(\zeta - \zeta_o)]\}$$

$$+ \lambda^4 \sin \lambda A \{- \cos A[2(\zeta - \zeta_o - A) + \sin 2(\zeta - \zeta_o - A)]$$

$$+ \sin(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o - 2A)$$

$$- \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A) + 2 \sin(\zeta - \zeta_o)$$

$$+ 2 \sin(\zeta - \zeta_o - A) - \sin A\}]$$

$$Y_{36} = \frac{1}{\lambda^2(\lambda^2 - 1)^2} \Big[- 3(1 - \cos \lambda A)[1 + \cos(\zeta - \zeta_o)]$$

$$+ 3\lambda \sin \lambda A[- (\zeta - \zeta_o - A)\cos(\zeta - \zeta_o) + \sin(\zeta - \zeta_o) + A]$$

$$+ \lambda^2 \{ (\zeta - \zeta_o)[- 3 \sin (\zeta - \zeta_o) + 3 \cos \lambda A \sin(\zeta - \zeta_o - A)$$

$$- 4 \sin A \cos \lambda A]$$

$$+ \cos \lambda A [- 6 + 7 \cos A + 3 \cos(\zeta - \zeta_o - A)$$

$$- 6 \cos(\zeta - \zeta_o) + 4A \sin A$$

$$+ 4 \cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)]$$

$$+ 3 \cos(\zeta - \zeta_o) - 4 \cos^2(\zeta - \zeta_o) - 1\}$$

$$+ \lambda^3 \sin \lambda A \{(\zeta - \zeta_o)[6 \cos(\zeta - \zeta_o) + 3 \cos(\zeta - \zeta_o - A)$$

$$+ 4 \cos A]$$

$$- 6A\cos(\zeta - \zeta_o) - 6\sin(\zeta - \zeta_o)$$

$$- 3\sin(\zeta - \zeta_o - A) - 6A - 4A\cos A + 13\sin A$$

$$- 2\sin 2(\zeta - \zeta_o - A)\cos A$$

$$+ 2\cos(\zeta - \zeta_o - 2A)\sin(\zeta - \zeta_o - A)$$

$$- 2\sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\Big\}$$

$$+ \lambda^4 \Big\{ (\zeta - \zeta_o)[\, 3\sin(\zeta - \zeta_o) - 3\cos\lambda A \sin(\zeta - \zeta_o - A)$$

$$+ 4\sin A\cos\lambda A]$$

$$+ \cos\lambda A \,[\, 3 - 3\cos A - 3\cos(\zeta - \zeta_o - A)$$

$$+ 3\cos(\zeta - \zeta_o) - 4A\sin A$$

$$+ 4\sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$- 4\sin^2(\zeta - \zeta_o)\Big\}$$

$$+ \lambda^5 \sin\lambda A \,\Big\{ - (\zeta - \zeta_o)[\, 3\cos(\zeta - \zeta_o) + 3\cos(- \zeta - \zeta_o - A)$$

$$+ 4\cos A]$$

$$+ 3A\cos(\zeta - \zeta_o) + 3\sin(\zeta - \zeta_o) - 5\sin A$$

$$+ 3\sin(\zeta - \zeta_o - A) + 3A + 4A\cos A$$

$$+ 2\sin 2(\zeta - \zeta_o - A)\cos A$$

$$+ 2\sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$- 2\cos(\zeta - \zeta_o - 2A)\sin(\zeta - \zeta_o - A)\Big\}\Big]$$

$$
\begin{aligned}
Z_{44} = \frac{1}{2(\lambda^2 - 1)^2} \Big[2 \Big\{ & (\zeta - \zeta_o - A) \sin A \cos \lambda A + \cos \zeta_o [\cos \zeta_o \\
& \qquad - \cos \lambda A \cos(\zeta_o + A)] \\
& + \cos \zeta [\cos \zeta - \cos \lambda A \cos(\zeta - A)] \Big\} \\
& + \sin \lambda A \Big\{ \cos A [- 2(\zeta - \zeta_o - A) - \sin 2(\zeta_o + A) \\
& \qquad + \sin 2(\zeta - A)] - 4 \sin A \\
& \qquad - \sin \zeta_o \cos(\zeta_o + A) \\
& \qquad + \sin(\zeta_o + A) \cos(\zeta_o + 2A) + \sin \zeta \cos(\zeta - A) \\
& \qquad - \sin(\zeta - A) \cos(\zeta - 2A) \Big\} \\
& + 2\lambda^2 \Big\{ - (\zeta - \zeta_o - A) \sin A \cos \lambda A \\
& \qquad + \sin \zeta_o [\sin \zeta_o - \cos \lambda A \sin(\zeta_o + A)] \\
& \qquad + \sin \zeta [\sin(\zeta - \cos \lambda A \sin(\zeta - A)] \Big\} \\
& + \lambda^3 \sin \lambda A \Big\{ \cos A [2(\zeta - \zeta_o - A) + \sin 2(\zeta_o + A) \\
& \qquad - \sin 2(\zeta - A)] + \sin \zeta_o \cos(\zeta_o + A) \\
& \qquad - \sin(\zeta_o + A) \cos(\zeta_o + 2A) - \sin \zeta \cos(\zeta - A) \\
& \qquad + \sin(\zeta - A) \cos(\zeta - 2A) \Big\} \Big]
\end{aligned}
$$

$$
\begin{aligned}
Z_{45} = \frac{1}{2(\lambda^2 - 1)^2} \Bigg(& -2 \sin \zeta_o \cos \zeta_o - 2 \sin \zeta \cos \zeta \\
& + \cos \lambda A \Big\{ 3 \sin \zeta_o \cos(\zeta_o + A) + 3 \sin \zeta \cos(\zeta - A) \\
& \qquad - \sin(\zeta_o + A)\cos(\zeta_o + 2A) - \sin(\zeta - A)\cos(\zeta - 2A) \\
& \qquad + 2 \sin A[\sin^2(\zeta - A) - \sin^2(\zeta_o + A)]\Big\} \\
& + \lambda \sin \lambda A \Big\{ - \sin A[\sin 2(\zeta_o + A) + \sin 2(\zeta - A)] \\
& \qquad + 2 \cos A[\sin^2(\zeta_o + A) - \sin^2(\zeta - A)]\Big\} \\
& + \lambda^2 \Big[2 \sin \zeta_o \cos \zeta_o + 2 \sin \zeta \cos \zeta \\
& \qquad + \cos \lambda A \Big\{ - \sin \zeta_o \cos(\zeta_o + A) - \sin \zeta \cos(\zeta - A) \\
& \qquad\qquad + \sin(\zeta_o + A)[\cos(\zeta_o + 2A) - 2 \cos \zeta_o] \\
& \qquad\qquad + \sin(\zeta - A)[\cos(\zeta - 2A) - 2 \cos \zeta] \\
& \qquad\qquad + 2 \sin A[\sin^2(\zeta_o + A) - \sin^2(\zeta - A)]\Big\}\Big] \\
& + \lambda^3 \sin \lambda A \Big\{ \sin A[\sin 2(\zeta_o + A) + \sin 2(\zeta - A)] \\
& \qquad + 2 \cos A[\sin^2(\zeta - A) - \sin^2(\zeta_o + A)]\Big\} \Bigg)
\end{aligned}
$$

$Z_{46} = Z_{45}$

$$Z_{55} = \frac{1}{2(\lambda^2 - 1)^2}\Big[2(\zeta - \zeta_o - A)\sin A\cos\lambda A$$

$$+ 2\sin\zeta_o[\sin\zeta_o - \cos\lambda A\sin(\zeta_o + A)]$$

$$+ 2\sin\zeta[\sin\zeta - \cos\lambda A\sin(\zeta - A)]$$

$$+ \lambda\sin\lambda A\{\cos A[-2(\zeta - \zeta_o - A) - \sin 2(\zeta - A)$$

$$+ \sin 2(\zeta_o + A)] - 4\sin A$$

$$+ \sin\zeta_o\cos(\zeta_o + A) - \sin\zeta\cos(\zeta - A)$$

$$- \sin(\zeta_o + A)\cos(\zeta_o + 2A) + \sin(\zeta - A)\cos(\zeta - 2A)\}$$

$$+ \lambda^2\{-2(\zeta - \zeta_o - A)\sin A\cos\lambda A$$

$$+ 2\cos\zeta_o[\cos\zeta_o - \cos\lambda A\cos(\zeta_o + A)]$$

$$+ 2\cos\zeta[\cos\zeta - \cos\lambda A\cos(\zeta - A)]\}$$

$$+ \lambda^3\sin\lambda A\{\cos A[2(\zeta - \zeta_o - A) + \sin 2(\zeta - A) - \sin 2(\zeta_o + A)]$$

$$- \sin\zeta_o\cos(\zeta_o + A) + \sin\zeta\cos(\zeta - A)$$

$$+ \sin(\zeta_o + A)\cos(\zeta_o + 2A) - \sin(\zeta - A)\cos(\zeta - 2A)\}\Big]$$

$Z_{56} = Z_{55}$

$X_{66} = Y_{11}$

$$
\begin{aligned}
Y_{66} = \frac{1}{\lambda^4(\lambda^2 - 1)^2} \Big[& 18\,(1 - \cos\lambda A) - 18\,\lambda A \sin\lambda A \\
& + 3\,\lambda^2 \{ 3(\zeta - \zeta_o)^2 (1 - \cos\lambda A) \\
& \qquad + \cos\lambda A [3A^2 + 4 - 8\cos(\zeta - \zeta_o)] - 4 \\
& \qquad + 8\cos(\zeta - \zeta_o) \} \\
& + 3\lambda^3 \sin\lambda A [\, 2(\zeta - \zeta_o)^3 - 3A(\zeta - \zeta_o)^2 \\
& \qquad + 8(\zeta - \zeta_o - A)\cos(\zeta - \zeta_o) + A^3 + 4A \\
& \qquad - 8\sin(\zeta - \zeta_o)] \\
& + 2\lambda^4 \{ -9(\zeta - \zeta_o)^2 (1 - \cos\lambda A) \\
& \qquad + (\zeta - \zeta_o) [\, 12\sin(\zeta - \zeta_o) - 12\cos\lambda A \sin(\zeta - \zeta_o - A) \\
& \qquad\qquad + 8\sin A \cos\lambda A] \\
& \qquad + \cos\lambda A [-9A^2 + 15 - 20\cos A - 12\cos(\zeta - \zeta_o - A) \\
& \qquad\qquad + 24\cos(\zeta - \zeta_o) - 8A\sin A \\
& \qquad\qquad - 8\cos(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)] + 5 \\
& \qquad - 12\cos(\zeta - \zeta_o) + 8\cos^2(\zeta - \zeta_o) \} \\
& + 2\lambda^5 \sin\lambda A \{ -6(\zeta - \zeta_o)^3 + 9A(\zeta - \zeta_o)^2 \\
& \qquad - 4(\zeta - \zeta_o)[\, 6\cos(\zeta - \zeta_o) + 3\cos(\zeta - \zeta_o - A) \\
& \qquad\qquad + 2\cos A]
\end{aligned}
$$

$$- 3A^3 + 15A + 24A\cos(\zeta - \zeta_o) + 24\sin(\zeta - \zeta_o)$$

$$- 32\sin A + 12\sin(\zeta - \zeta_o - A) + 8A\cos A$$

$$+ 4\sin 2(\zeta - \zeta_o - A)\cos A$$

$$- 4\cos(\zeta - \zeta_o - 2A)\sin(\zeta - \zeta_o - A)$$

$$+ 4\sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)\}$$

$$+ \lambda^6 \{ 9(\zeta - \zeta_o)^2(1 - \cos\lambda A)$$

$$+ (\zeta - \zeta_o)[- 24\sin(\zeta - \zeta_o) + 24\cos\lambda A\sin(\zeta - \zeta_o - A)$$

$$- 16\sin A\cos\lambda A]$$

$$+ \cos\lambda A[9A^2 - 24 + 24\cos A + 24\cos(\zeta - \zeta_o - A)$$

$$- 24\cos(\zeta - \zeta_o) + 16A\sin A$$

$$- 16\sin(\zeta - \zeta_o)\sin(\zeta - \zeta_o - A)]$$

$$+ 16\sin^2(\zeta - \zeta_o)\}$$

$$+ \lambda^7 \sin\lambda A\{6(\zeta - \zeta_o)^3 - 9A(\zeta - \zeta_o)^2$$

$$+8(\zeta - \zeta_o)[3\cos(\zeta - \zeta_o) + 3\cos(\zeta - \zeta_o - A)$$

$$+ 2\cos A]$$

$$+ 3A^2 - 24A\cos(\zeta - \zeta_o) - 24\sin(\zeta - \zeta_o)$$

$$+ 32\sin A - 24\sin(\zeta - \zeta_o - A) - 24A$$

$$- 16A\cos A - 8\sin 2(\zeta - \zeta_o - A)\cos A$$

$$- 8 \sin(\zeta - \zeta_o)\cos(\zeta - \zeta_o - A)$$

$$+ 8 \cos(\zeta - \zeta_o - 2A)\sin(\zeta - \zeta_o - A)\}\Big]$$

$$Z_{66} = Z_{55}$$

The indeterminacies occuring in the above equations, when $\lambda = 0$ and when $\lambda = 1$, are resolved by direct application of L'Hospital's rule.

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Lower bounds are obtained for the covariances of the orbital elements of an initially circular orbit due to omissions of harmonics of a given degree in the expansion of the earth's gravity potential. This is accomplished by applying the methods of linear estimation theory to the surface gravimetry results of Kaula, extended to satellite altitudes. Numerical results are presented for the case of a 100-n mi geocentric orbit.

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